First-order partial derivatives of functions with two variables

Dr. Aspriha Peters

Engineering math blog

https://www.engineeringmathgeek.com/

engineeringmathgeek@gmail.com

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1 First-order partial derivatives of functions with two variables

Suppose z is a function of two variables x and y. Let me choose $z = x + y$. Now I want to get the first-order partial derivative of z with respect to both x and y.

The mathematical symbol of the first-order partial derivative of z with respect to x is $\frac{\partial z}{\partial x}$. Also, the mathematical symbol of the first-order partial derivative of z with respect to y is $\frac{\partial z}{\partial y}$.

Next, I'll determine the value of $\frac{\partial z}{\partial x}$. This means I'll differentiate z partially with respect to x. At that time, the other variable y will act as a constant. Also, for partial differentiation, the rules are the same as ordinary differentiation.

So $\frac{\partial z}{\partial x}$ will be

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x + y).$$

This gives

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(y)\\ &= 1+0\\ &= 1.\end{aligned}$$

In the same way $\frac{\partial z}{\partial y}$ will be

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x+y).$$

Therefore I can say that

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(y)\\ &= 0+1\\ &= 1.\end{aligned}$$

And there are many more examples like that.So now I'll give you some examples.

1.1 Examples of the first-order partial derivatives of functions with two variables

Disclaimer: None of these examples is mine. I have chosen these from some book or books. I have also given the due reference at the end of the post.

So here are the examples.

1.1.1 Example 1

According to [Stroud and Booth (2013)]* "Find all first and second partial derivatives of the following: $z = 4x^3 - 5xy^2 + 3y^3$."

Solution

In this post, I'll only do the first-order partial derivatives. Soon I'll write another post on second-order partial derivatives.

So, here it is.

Here the given function is $z = 4x^3 - 5xy^2 + 3y^3$.

I have to get first-order partial derivatives of z. That means I need to find out the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. I'll start with $\frac{\partial z}{\partial x}$.

First of all, I'll differentiate z partially with respect to x.This gives

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(4x^3 - 5xy^2 + 3y^3\right) \\ &= \frac{\partial}{\partial x}\left(4x^3\right) - \frac{\partial}{\partial x}\left(5xy^2\right) + \frac{\partial}{\partial x}\left(3y^3\right).\end{aligned}$$

Now for $\frac{\partial z}{\partial x}$, the variable y will act as a constant.

So it gives

$$\begin{aligned}\frac{\partial z}{\partial x} &= 12x^2 - 5y^2 + 0 \\ &= 12x^2 - 5y^2.\end{aligned}$$

Now I'll get the value of $\frac{\partial z}{\partial y}$.

Here again I'll differentiate z partially with respect to y. This gives

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(4x^3 - 5xy^2 + 3y^3\right) \\ &= \frac{\partial}{\partial y}\left(4x^3\right) - \frac{\partial}{\partial y}\left(5xy^2\right) + \frac{\partial}{\partial y}\left(3y^3\right).\end{aligned}$$

Now for $\frac{\partial z}{\partial y}$, the variable x will act as a constant.

So it gives

$$\begin{aligned}\frac{\partial z}{\partial y} &= 0 - 5x(2y) + 9y^2 \\ &= -10xy + 9y^2.\end{aligned}$$

Hence I can conclude that the first partial derivatives of the function z are $\frac{\partial z}{\partial x} = 12x^2 - 5y^2$ and $\frac{\partial z}{\partial y} = -10xy + 9y^2$.

This is the answer to this example.

Now I'll go to the next example.

1.1.2 Example 2

According to [Stroud and Booth (2013)]* "Find all first and second partial derivatives of the following: $z = e^{x^2-y^2}$."

Solution

Like my first example, here also I'll only do the first-order partial derivatives. Soon I'll write another post on second-order partial derivatives.

So, here it is.

Here the given function is $z = e^{x^2-y^2}$.

I have to get first-order partial derivatives of z. That means I need to find out the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

I'll start with $\frac{\partial z}{\partial x}$.

First of all, I'll differentiate z partially with respect to x. This gives

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(e^{x^2-y^2}\right) \\ &= e^{x^2-y^2}\frac{\partial}{\partial x}\left(x^2-y^2\right). \end{aligned}$$

Now for $\frac{\partial z}{\partial x}$, the variable y will act as a constant.

So it gives

$$\begin{aligned} \frac{\partial z}{\partial x} &= e^{x^2-y^2}(2x-0) \\ &= 2xe^{x^2-y^2}. \end{aligned}$$

Now I'll get the value of $\frac{\partial z}{\partial y}$.

Here again, I'll differentiate z partially with respect to y.

This gives

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}\left(e^{x^2-y^2}\right) \\ &= e^{x^2-y^2}\frac{\partial}{\partial y}\left(e^{x^2-y^2}\right). \end{aligned}$$

Now for $\frac{\partial z}{\partial y}$, the variable x will act as a constant.

So it gives

$$\begin{aligned}\frac{\partial z}{\partial y} &= e^{x^2-y^2}\left(e^{0-2y}\right) \\ &= -2ye^{x^2-y^2}.\end{aligned}$$

Hence I can conclude that the first partial derivatives of the function z are $\frac{\partial z}{\partial x} = 2xe^{x^2-y^2}$ and $\frac{\partial z}{\partial y} = -2ye^{x^2-y^2}$.

Hence I can conclude that this is the answer to the given example.

Now I am going to give two more examples. Also, I don't have a clue about the references to these two examples.

1.1.3 Example 3

Find all of the first order partial derivatives for the following functions:(c) $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[3]{s^4}$.

Solution

Here the given function is $h(s,t) = t^7 \ln(s^2) + \frac{9}{t^3} - \sqrt[7]{s^4}$. As I can see, h is a function of two variables s and t. And I have to get first-order partial derivatives of h.

So that means I need to find out the values of $\frac{\partial h}{\partial s}$ and $\frac{\partial h}{\partial t}$. But before that, I'll simplify the function h.

Step 1

For example, I can write $\ln(s^2)$ as $2\ln s$. And this is because of $\ln a^n = n \ln a$.

Also, I can rewrite $\frac{9}{t^3}$ as $9t^{-3}$.

Again, I can rewrite $\sqrt[7]{s^4}$ as $s^{4/7}$ since $\sqrt[m]{a^n} = a^{n/m}$.

So this means the function h becomes

$$h = 2t^7 \ln s + 9t^{-3} - s^{4/7}.$$

Now I'll differentiate h partially with respect to s and t.

Step 2

If I differentiate h partially with respect to s, it gives

$$\frac{\partial h}{\partial s} = \frac{\partial}{\partial s}(2t^7 \ln s + 9t^{-3} - s^{4/7}).$$

Now for $\frac{\partial h}{\partial s}$, the variable t will act as a constant. So it gives

$$\frac{\partial h}{\partial s} = 2t^7 \frac{1}{s} + 0 - \frac{4}{7} s^{4/7-1}.$$

Next, I'll simplify it to get

$$\frac{\partial h}{\partial s} = \frac{2t^7}{s} - \frac{4}{7} s^{-3/7}.$$

Now I'll get the value of $\frac{\partial h}{\partial t}$. So I'll differentiate h partially with respect to t. This gives

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial t}(2t^7 \ln s + 9t^{-3} - s^{4/7}).$$

Now for $\frac{\partial h}{\partial t}$, the variable s will act as a constant. So it gives

$$\frac{\partial h}{\partial t} = 2 \ln s(7t^{7-1}) + 9(-3)t^{-3-1} - 0.$$

Then I'll simplify it to get

$$\frac{\partial h}{\partial t} = 14t^6 \ln s - 27t^{-4}.$$

Hence I can conclude that the first partial order derivatives of the function h are $\frac{\partial h}{\partial s} = \frac{2t^7}{s} - \frac{4}{7} s^{-3/7}$ and $\frac{\partial h}{\partial t} = 14t^6 \ln s - 27t^{-4}$.

Therefore I can say that this is the answer to the given example.

Now I am going to give another example.

1.1.4 Example 4

Find all of the first order partial derivatives for the following functions:(d) $\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}$.

Solution

Here the given function is $f(x,y) = \cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}$.

I have to get first-order partial derivatives of $f(x,y)$. That means I need to find out the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. I'll start with $\frac{\partial f}{\partial x}$.

First of all, I'll differentiate f partially with respect to x.

Step 1

And this gives

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}\right).$$

Now for $\frac{\partial z}{\partial x}$, the variable y will act as a constant. Also, I can see the function f is a product of two functions $\cos\left(\frac{4}{x}\right)$ and $e^{(x^2y-5y^3)}$. So in this case, I'll use the product rule for differentiation to differentiate the function f.

Therefore it will be

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\cos\left(\frac{4}{x}\right)\right) e^{(x^2y-5y^3)} + \cos\left(\frac{4}{x}\right)\frac{\partial}{\partial x}(e^{(x^2y-5y^3)}).$$

Next, I'll simplify it to get

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{(x^2y-5y^3)}(-\sin\left(\frac{4}{x}\right)\frac{\partial}{\partial x}\left(\frac{4}{x}\right)) \\ &\quad + \cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}\frac{\partial}{\partial x}(x^2y-5y^3).\end{aligned}$$

Now I'll do further partial differentiation to get

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{(x^2y-5y^3)}(-\sin\left(\frac{4}{x}\right)\left(-\frac{4}{x^2}\right)) \\ &\quad + \cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}(2xy-0).\end{aligned}$$

Then I'll simplify it to get

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{4}{x^2}\sin\left(\frac{4}{x}\right) e^{(x^2y-5y^3)} \\ &\quad + 2xy\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}.\end{aligned}$$

Now I'll get the value of $\frac{\partial f}{\partial y}$.

Step 2

Here again, I'll differentiate f partially with respect to y.

This gives

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}\right).$$

Now for $\frac{\partial f}{\partial y}$, the variable x will act as a constant. Then it gives

$$\frac{\partial f}{\partial y} = \cos\left(\frac{4}{x}\right) \frac{\partial}{\partial y}(e^{(x^2y-5y^3)}).$$

Next, I'll simplify it to get

$$\begin{aligned}\frac{\partial f}{\partial y} &= \cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)} \\ &\quad \times \frac{\partial}{\partial y}(x^2y-5y^3).\end{aligned}$$

So this becomes

$$\frac{\partial f}{\partial y} = \cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)} \times (x^2-15y^2).$$

Then I can rearrange them to get

$$\frac{\partial f}{\partial y} = (x^2-15y^2)\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}.$$

Hence I can conclude that the first partial order derivatives of the function f are $\frac{\partial f}{\partial x} = \frac{4}{x^2}\sin\left(\frac{4}{x}\right) e^{(x^2y-5y^3)} + 2xy\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}$ and $\frac{\partial f}{\partial y} = (x^2-15y^2)\cos\left(\frac{4}{x}\right) e^{(x^2y-5y^3)}$.

Therefore I can conclude that this is the answer to the given example.

Dear friends, this is the end of today's post. Thank you very much for reading this. Please let me know how you feel about it. Soon I will be back again with a new post. Till then, bye, bye!!

References

[Stroud and Booth (2013)] *: Engineering mathematics, Industrial Press, Inc.; 7th Edition (March 8, 2013), Chapter: Partial differentiation 1, Test exercise 14, p. 749, Q. No. 1(a) (Example 1); Q. No. 1(c) (Example 2).