First shift theorem in Laplace transform

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Example 1

According to Stroud and Booth (2011)* "Determine the Laplace transform of the following function: $e^{-2t}\cos 5t$."

Solution

Now here I have to find out the Laplace transform of the function $e^{-2t}\cos 5t$. As I have said earlier in this post, the first shift theorem of Laplace transform says if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$. And in this example, $f(t)$ is $\cos 5t$. And, e^{-at} is e^{-2t}. So that means $a = 2$. Also, I already know the Laplace transform of $\cos 5t$ from the standard formulas in Laplace transform. That is

$$\mathcal{L}\{\cos 5t\} = \frac{s}{(s)^2 + (5)^2}.$$

Therefore according to the first shift theorem, $\mathcal{L}\{e^{-2t}\cos 5t\}$ will be

$$\mathcal{L}\{e^{-2t}\cos 5t\} = \frac{(s+2)}{(s+2)^2 + (5)^2}.$$

Now I'll simplify it to get the final value of $\mathcal{L}\{e^{-2t}\cos 5t\}$. Thus it will be

$$\begin{aligned}\mathcal{L}\{e^{-2t}\cos 5t\} &= \frac{(s+2)}{(s^2+4s+4)+25} \\ &= \frac{(s+2)}{s^2+4s+29}.\end{aligned}$$

Hence I can conclude that the Laplace transform of the function $e^{-2t}\cos 5t$ is $\frac{(s+2)}{s^2+4s+29}$. And this is the answer to this example.Now I'll give you another example.

Example 2

According to Stroud and Booth (2011)* "Determine the Laplace transform of the following function: $e^{4t}\cos 2t$."

Solution

Here I have to find out the Laplace transform of the function $e^{4t}\cos 2t$. Now as I have said earlier in this post, the first shift theorem of Laplace transform says if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$. In this example, $f(t)$ is $\cos 2t$. And, e^{-at} is e^{4t}. That means $a = -4$. I already know the Laplace transform of $\cos 2t$ from the standard formulas in Laplace transform. That is

$$\mathcal{L}\{\cos 2t\} = \frac{s}{(s)^2+(2)^2}.$$

Therefore according to the first shift theorem, $\mathcal{L}\{e^{4t}\cos 2t\}$ will be

$$\mathcal{L}\{e^{4t}\cos 2t\} = \frac{(s-4)}{(s-4)^2+(2)^2}.$$

Now I'll simplify it to get the final value of $\mathcal{L}\{e^{4t}\cos 2t\}$. Thus it will be

$$\begin{aligned}\mathcal{L}\{e^{4t}\cos 2t\} &= \frac{(s-4)}{(s^2-8s+16)+4} \\ &= \frac{(s-4)}{s^2-8s+20}.\end{aligned}$$

Hence I can conclude that the Laplace transform of the function $e^{4t}\cos 2t$ is $\frac{(s-4)}{s^2-8s+20}$.

This is the answer to this example.

Example 3

According to Stroud and Booth (2011)* "Determine the Laplace transform of the following function: $e^{3t}(t^2+4)$."

Solution

Here I have to find out the Laplace transform of the function $e^{3t}(t^2+4)$. I can also rewrite it as a sum of two functions. Then it will be

$$e^{3t}(t^2+4) = t^2e^{3t} + 4e^{3t}.$$

So as a first step, I'll find out the Laplace transform of the function t^2e^{3t}. Next, I'll find out the Laplace transform of the function $4e^{3t}$. At the end, I will add these two. So here goes the first step.

Step 1

Now as I have said earlier, the first shift theorem of Laplace transform says if $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$. Thus for the function t^2e^{3t}, $f(t)$ is t^2. And, e^{-at} is e^{3t}. That means $a = -3$.

As per the standard formulas in Laplace transform, the Laplace transform of the function t^2 is

$$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}}.$$

Therefore according to the first shift theorem, $\mathcal{L}\{t^2e^{3t}\}$ will be

$$\mathcal{L}\{t^2e^{3t}\} = \frac{2!}{(s-3)^{2+1}}.$$

This means

$$\mathcal{L}\{t^2e^{3t}\} = \frac{2}{(s-3)^3}. \tag{1}$$

Now I'll go to the second step.

Step 2

Here I'll find out the Laplace transform of the function $4e^{3t}$.

Thus for the function $4e^{3t}$, $f(t)$ is 4. And, e^{-at} is e^{3t}. That means $a = -3$. As per the standard formulas in Laplace transform, the Laplace transform of the function 4 is

$$\mathcal{L}\{4\} = \frac{4}{s}.$$

Therefore according to the first shift theorem, $\mathcal{L}\{4e^{3t}\}$ will be

$$\mathcal{L}\{4e^{3t}\} = \frac{4}{s-3}. \tag{2}$$

Also I have mentioned earlier that

$$\mathcal{L}\{e^{3t}(t^2+4)\} = \mathcal{L}\{t^2e^{3t}\} + \mathcal{L}\{4e^{3t}\}.$$

Therefore I'll add equations (1) and (2) to get the Laplace transform of the function $e^{3t}(t^2+4)$. Thus it will be

$$\mathcal{L}\{t^2e^{3t}\} + \mathcal{L}\{4e^{3t}\} = \frac{2}{(s-3)^3} + \frac{4}{s-3}.$$

Next, I'll simplify it to get the final answer. Hence it will look like

$$\begin{aligned}
\mathcal{L}\{e^{3t}(t^2+4)\} &= \frac{2+4(s-3)^2}{(s-3)^3} \\
&= \frac{2+4(s^2-6s+9)}{(s-3)^3} \\
&= \frac{2+4s^2-24s+36}{(s-3)^3} \\
&= \frac{4s^2-24s+38}{(s-3)^3}.
\end{aligned}$$

Hence I can conclude that the Laplace transform of the function $e^{3t}(t^2+4)$ is $\frac{4s^2-24s+38}{(s-3)^3}$.

This is the answer to this example.

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*Reference: K. A. Stroud and Dexter J. Booth (2011): Advanced engineering mathematics, Industrial Press, Inc.; 5th Edition (March 8, 2011), Chapter: Laplace transform 1, Further problems 2, p. 90, Q. No. 1(a) (Example 2); Q. No. 1(d) (Example 3).