Second-order homogeneous ODE with complex conjugate roots

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Example 1

According to Kreyszig (2005)* Find the general solution of the following: $y'' + 9\pi^2 y = 0$.

Solution

Here the given second-order differential equation is

$$y'' + 9\pi^2 y = 0.$$

Therefore the auxiliary equation (characteristic equation) will be

$$m^2 + 9\pi^2 = 0.$$

Now I'll solve this equation to get the values of m. So I can also rewrite this equation as

$$\begin{aligned} m^2 + 9\pi^2 &= 0 \\ m^2 &= -9\pi^2. \end{aligned}$$

This means

$$m = \pm 3\pi j.$$

So I can say $m = 3\pi j$ or $m = -3\pi j$. That shows the given differential equation has complex conjugate roots. Therefore the general solution of the given differential equation is

$$y = (C_1 \cos 3\pi x + C_2 \sin 3\pi x).$$

This is the solution to the given equation. Next, I'll show you another example.

Example 2

According to Kreyszig (2005)* Solve the following initial value problem: $20y'' + 4y' + y = 0; y(0) = 3.2, y'(0) = 0$.

Solution

Here the given second-order differential equation is

$$20y'' + 4y' + y = 0.$$

Also I know the initial values of y and y' at $x = 0$. That is $y(0) = 3.2, y'(0) = 0$. My task is to solve this initial value problem. For that, I'll start with the general solution of this equation.

Step 1

Therefore the auxiliary equation (characteristic equation) will be

$$20m^2 + 4m + 1 = 0.$$

Now I'll solve this equation to get the values of m. Thus the values of m will be

$$m = \frac{-4 \pm \sqrt{(4)^2 - 4(20)(1)}}{2(20)}.$$

Next, I'll simplify it to get

$$\begin{aligned} m &= \frac{-4 \pm \sqrt{16 - 80}}{40} \\ &= \frac{-4 \pm \sqrt{-64}}{40} \\ &= \frac{-4 \pm 8j}{40}. \end{aligned}$$

This means

$$m = \frac{-1 \pm 2j}{10}.$$

Now this gives $m = -0.1 \pm 0.2j$. That shows the given differential equation has complex conjugate roots. Therefore the general solution of the given differential equation is

$$y = e^{-0.1x}(C_1 \cos 0.2x + C_2 \sin 0.2x). \tag{1}$$

Now I'll get the values of the constants C_1 and C_2.

Step 2

Also it is given that $y(0) = 3.2, y'(0) = 0$. This means at $x = 0, y = 3.2$ and $y' = 0$. So, first I'll put $x = 0$ and $y = 3.2$ in equation (1). Therefore it will be

$$3.2 = e^{-0.1(0)}(C_1 \cos 0.2(0) + C_2 \sin 0.2(0)).$$

That means

$$3.2 = e^0(C_1 \cos 0 + C_2 \sin 0).$$

Now it is known that $\cos 0 = 1$ and $\sin 0 = 0$. Therefore it becomes

$$\begin{aligned} 3.2 &= 1(C_1.1 + C_2.0) \\ 3.2 &= C_1 + 0. \end{aligned}$$

This gives $C_1 = 3.2$. Now I'll differentiate equation (1) with respect to x. This gives

$$y' = -0.1e^{-0.1x}(C_1 \cos 0.2x + C_2 \sin 0.2x) + e^{-0.1x}(-0.2C_1 \sin 0.2x + 0.2C_2 \cos 0.2x). \tag{2}$$

Next, I'll put $x = 0, y' = 0$ and $C_1 = 3.2$ in equation (2). This means

$$\begin{aligned} 0 &= -0.1e^{-0.1(0)}(3.2 \cos 0.2(0) + C_2 \sin 0.2(0)) \\ &\quad +e^{-0.1(0)}(-0.2(3.2) \sin 0.2(0) + 0.2C_2 \cos 0.2(0)). \end{aligned}$$

Now I'll simplify it to get

$$
\begin{aligned}
0 &= -0.1e^0(3.2\cos 0 + C_2\sin 0) + e^0(-0.2(3.2)\sin 0 + 0.2C_2\cos 0) \\
&= -0.1(1)(3.2.1 + C_2.0) + 1(-0.2(3.2).0 + 0.2C_2.1) \\
&= -0.1(3.2 + 0) + (-0 + 0.2C_2) \\
&= -0.32 + 0.2C_2.
\end{aligned}
$$

This gives $C_2 = \dfrac{0.32}{0.2} = 1.6$. Now I'll put back $C_1 = 3.2$ and $C_2 = 1.6$ in equation (1). This will give $y = e^{-0.1x}(3.2\cos 0.2x + 1.6\sin 0.2x)$. Thus the solution of the differential equation is

$$y = e^{-0.1x}(3.2\cos 0.2x + 1.6\sin 0.2x).$$

Hence this is the solution to the given equation.

Kreyszig, E. (2005)*: Advanced Engineering Mathematics: International Edition, John Wiley and Sons, 9th Edition, 29th December 2005, Chapter 2, Second-order linear ODEs, p. 59, Problem set 2.2, Q. 1(Example 11), Q. 2 (Example 29).