Shifted data problem — Laplace transform

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Example 1

According to Kreyszig (2005)* "Solve the following initial value problem by Laplace transform:

$$y^{'} - 6y = 0, \qquad y(2) = 4.''$$

Solution

Here the given differential equation with initial condition is

$$y^{'} - 6y = 0, \qquad y(2) = 4.$$

Now I have to solve this equation using Laplace transform. As I have mentioned above, here $t_0 = 2$. So I can set $t = \widetilde{t} + 2$. Then the Laplace transform of the given equaiton will be

$$\begin{aligned} \mathcal{L}\{y^{'} - 6y\} &= \mathcal{L}\{0\} \\ (s\widetilde{Y} - \widetilde{Y}_0) - 6\widetilde{Y} &= 0. \end{aligned}$$

Now I already know from the given equation that $\widetilde{Y}_0 = y(0) = 4$. Therefore I'll simplify this equation to get

$$\begin{aligned} s\widetilde{Y} - 4 - 6\widetilde{Y} &= 0 \\ \widetilde{Y}(s-6) &= 4 \\ \widetilde{Y} &= \frac{4}{s-6}. \end{aligned}$$

Next, I'll find out the value of $\widetilde{y}$ as

$$\begin{aligned} \widetilde{y} &= \mathcal{L}^{-1}\left\{\frac{4}{s-6}\right\} \\ &= 4e^{6\widetilde{t}}. \end{aligned}$$

As I already know, $\widetilde{t} = t - 2$. So it will be $\mathcal{L}^{-1}\left\{\dfrac{4}{s-6}\right\} = 4e^{6(t-2)}$.

Hence I can conclude that the solution of the differential equation will be

$$y = 4e^{6(t-2)}.$$

This is the answer to this example.

Now I'll give another example.

Example 2

According to Kreyszig (2005)* "Solve the following initial value problem by Laplace transform:

$$y'' - 2y' - 3y = 0, \qquad y(1) = -3, \qquad y'(1) = -17."$$

Solution

Here the given differential equation with initial condition is

$$y'' - 2y' - 3y = 0, \qquad y(1) = -3, \qquad y'(1) = -17.$$

Now I have to solve this equation using Laplace transform. As I have mentioned above, here $t_0 = 1$. So I can set $t = \widetilde{t} + 1$. Then the Laplace transform of the given equaiton will be

$$\begin{aligned}\mathcal{L}\{y^{''} - 2y^{'} - 3y\} &= \mathcal{L}\{0\} \\ (s^2\widetilde{Y} - s\widetilde{Y}_0 - \widetilde{Y}_0^{'}) - 2(s\widetilde{Y} - \widetilde{Y}_0) - 3\widetilde{Y} &= 0.\end{aligned}$$

Now I already know from the given equation that $\widetilde{Y}_0 = y(0) = -3, \widetilde{Y}_0^{'} = y^{'}(0) = -17$. Therefore I'll simplify this equation to get

$$\begin{aligned}(s^2\widetilde{Y} - s(-3) - (-17)) - 2s\widetilde{Y} + 2(-3) - 3\widetilde{Y} &= 0 \\ s^2\widetilde{Y} + 3s + 17 - 2s\widetilde{Y} - 6 - 3\widetilde{Y} &= 0 \\ \widetilde{Y}(s^2 - 2s - 3) + 3s + 11 &= 0 \\ \widetilde{Y} &= \frac{-3s - 11}{s^2 - 2s - 3}.\end{aligned}$$

Now I'll factorise $(s^2 - 2s - 3)$ as

$$s^2 - 2s - 3 = (s + 1)(s - 3).$$

Thus I can say

$$\widetilde{Y} = \frac{-3s - 11}{(s + 1)(s - 3)}.$$

Next, I'll find out the value of $\widetilde{y}$. Hence it will be

$$\widetilde{y} = \mathcal{L}^{-1}\left\{\frac{-3s - 11}{(s + 1)(s - 3)}\right\}.$$

Now I'll use partial fractions to get the inverse Laplace transform of this expression. So I'll rewrite the expression $\dfrac{-3s - 11}{(s + 1)(s - 3)}$ as a sum of two fractions like

$$\frac{-3s - 11}{(s + 1)(s - 3)} = \frac{A}{s + 1} + \frac{B}{s - 3}.$$

Now my task is to find the values of A and B. Here the denominators on the right-hand side have only s, not s^2 or any other higher power of s. So I can use the cover-up rule to find the inverse Laplace transform.

As I can see, the value of A is the coefficient of $\dfrac{1}{s + 1}$. So, by cover up rule, it will be

$$A = \lim_{s \to -1} \frac{-3s - 11}{s - 3} = \frac{(-3)(-1) - 11}{-1 - 3} = \frac{3 - 11}{-4} = \frac{-8}{-4} = 2.$$

In the same way, now I'll get the value of B. So here the value of B is the coefficient of $\frac{1}{s-3}$.

$$B = \lim_{s \to 3} \frac{-3s - 11}{s+1} = \frac{-3(3) - 11}{3+1} = \frac{-9-11}{4} = \frac{-20}{4} = -5.$$

Therefore I can say

$$\frac{-3s-11}{(s+1)(s-3)} = \frac{2}{s+1} - \frac{5}{s-3}.$$

Now I'll find out the value of $\widetilde{y}$ as

$$\begin{aligned} \widetilde{y} &= \mathcal{L}^{-1}\left\{\frac{2}{s+1} - \frac{5}{s-3}\right\} \\ &= 2e^{-\widetilde{t}} - 5e^{3\widetilde{t}}. \end{aligned}$$

As I already know, $\widetilde{t} = t - 1$. So it will be $\mathcal{L}^{-1}\left\{\frac{-3s-11}{(s+1)(s-3)}\right\} = 2e^{-(t-1)} - 5e^{3(t-1)}$.

Hence I can conclude that the solution of the differential equation will be

$$y = 2e^{-(t-1)} - 5e^{3(t-1)}.$$

This is the answer to this example.

Kreyszig, E. (2005): Advanced Engineering Mathematics: International Edition, John Wiley and Sons, 9th Edition, 29th December 2005, Chapter 6, Laplace transform, p. 233, Problem set 6.2, Example 1(Q. 21), Example 2 (Q. 22).