Partial fractions of equal degree expressions

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Example 1

Write in partial fraction form: $\dfrac{7x-5}{3x+1}$.

Solution

Here the given expression is $\dfrac{7x-5}{3x+1}$. As I can see, both the numerator and the denominator has the degree 1. So this is an equal degree expression. Now in order to write this expression in partial fraction form, I have to remove x from the top.

$$\frac{7x-5}{3x+1} = \frac{\frac{7}{3}(3x+1) - 5 - 1(\frac{7}{3})}{3x+1}.$$

Next, I'll simplify it to get

$$\begin{aligned}\frac{7x-5}{3x+1} &= \frac{\frac{7}{3}(3x+1)}{3x+1} + \frac{-(\frac{22}{3})}{3x+1} \\ &= \frac{7}{3} - \frac{22}{3(3x+1)}.\end{aligned}$$

Now I can say $\dfrac{7}{3} - \dfrac{22}{3(3x+1)}$ is the partial fraction of $\dfrac{7x-5}{3x+1}$. Next, I'll give other examples where both the numerator and the denominator have the degree 2.

Example 2

Write in partial fraction form: $\dfrac{2x^2+x-4}{x^2-3x+2}$.

Solution

Here the given expression is $\dfrac{2x^2+x-4}{x^2-3x+2}$. Since both the numerator and the denominator has the degree 2, this is an equal degree expression. In order to write this expression in partial fraction form, I have to remove both x and x^2 from the top. So I can write it as

$$\frac{2x^2+x-4}{x^2-3x+2} = \frac{2(x^2-3x+2)+7x-8}{x^2-3x+2}.$$

Then I'll simplify it to get

$$\begin{aligned}\frac{2x^2+x-4}{x^2-3x+2} &= \frac{2(x^2-3x+2)}{x^2-3x+2} + \frac{7x-8}{x^2-3x+2}\\ &= 2 + \frac{7x-8}{x^2-3x+2}.\end{aligned}$$

Now I have two components of this expression: one is 2 and the other is $\dfrac{7x-8}{x^2-3x+2}$. I cannot decompose 2 further. But I can still break down $\dfrac{7x-8}{x^2-3x+2}$. To start with, first I'll factorize (x^2-3x+2). Thus it will be

$$x^2-3x+2 = (x-1)(x-2).$$

Next, I'll rewrite $\dfrac{7x-8}{x^2-3x+2}$ as $\dfrac{7x-8}{(x-1)(x-2)}$. Let me assume

$$\frac{7x-8}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

where A, B are constants. Now my task is to find out the values of A and B. In order to do that, I'll simplify this expression as

$$\begin{aligned}\frac{7x-8}{(x-1)(x-2)} &= \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}.\\ &= \frac{(A+B)x-(2A+B)}{(x-1)(x-2)}.\end{aligned}$$

Now I can compare coefficients on both sides. And I get two different equations as

$$A + B = 7 \tag{1}$$

and

$$2A + B = 8. \tag{2}$$

Next, I'll solve equations (1) and (2) to get the values of A and B. Now from equation (1), I can say, $A = 7 - B$. Then I substitute this value of A in equation (2) to get the value of B. So this gives

$$B = 6.$$

Thus for $B = 6$, $A = 1$. Therefore the partial fraction of $\frac{7x-8}{(x-1)(x-2)}$ is $\frac{1}{x-1} + \frac{6}{x-2}$. Hence I can say that the partial fraction form of $\frac{2x^2+x-4}{x^2-3x+2}$ is $2 + \frac{1}{x-1} + \frac{6}{x-2}$. And this is the answer to the given example. Now I'll give some other examples.

Disclaimer: None of these examples is mine. I have chosen these from some book or books. I have also given the due reference at the end of the post.

Example 3

According to Stroud and Booth (2013)*, "Express each of the following in partial fractions: $\frac{4x^2-47x+141}{x^2-13x+40}$."

Solution

The given expression is $\frac{4x^2-47x+141}{x^2-13x+40}$. As I can see, both the numerator and the denominator has the degree 2. So this is an equal degree expression. First of all, I'll remove both x and x^2 from the top. As example 1, here also I'll start in the same way. So I can write this expression as

$$\frac{4x^2-47x+141}{x^2-13x+40}=\frac{4(x^2-13x+40)+5x-19}{x^2-13x+40}.$$

Then I'll simplify it to get

$$\begin{aligned}\frac{4x^2-47x+141}{x^2-13x+40} &= \frac{4(x^2-13x+40)}{x^2-13x+40}+\frac{5x-19}{x^2-13x+40}\\ &= 4+\frac{5x-19}{x^2-13x+40}\end{aligned}$$

Now I have two components of this expression: one is 4 and the other is $\frac{5x-19}{x^2-13x+40}$.

I cannot decompose 4 further. But I can still break down $\frac{5x-19}{x^2-13x+40}$. To start with, first I'll factorize $(x^2-13x+40)$. Thus it will be

$$x^2-13x+40=(x-5)(x-8).$$

Next, I'll rewrite $\frac{5x-19}{x^2-13x+40}$ as $\frac{5x-19}{(x-5)(x-8)}$.Let me assume

$$\frac{5x-19}{(x-5)(x-8)}=\frac{A}{x-5}+\frac{B}{x-8}$$

where A, B are constants. Now my task is to find out the values of A and B. In order to do that, I'll simplify this expression as

$$\begin{aligned}\frac{5x-19}{(x-5)(x-8)} &= \frac{A(x-8)+B(x-5)}{(x-5)(x-8)}\\ &= \frac{(A+B)x+(-8A-5B)}{(x-5)(x-8)}.\end{aligned}$$

Now I can compare coefficients on both sides. And I get two different equations as

$$A+B=5 \tag{3}$$

and

$$-8A-5B=-19. \tag{4}$$

Next, I'll solve equations (3) and (4) to get the values of A and B. Now I'll multiply equation (3) with 8 and add it to the equation (4). And this gives

$$\begin{aligned}3B &= 21\\ B &= 7.\end{aligned}$$

Next, I'll substitute this value of B in equation (3) to get

$$A = -2.$$

Therefore the partial fraction of $\dfrac{5x-19}{(x-5)(x-8)}$ is $-\dfrac{2}{x-5}+\dfrac{7}{x-8}$. Hence I can say that the partial fraction form of $\dfrac{4x^2-47x+141}{x^2-13x+40}$ is $4-\dfrac{2}{x-5}+\dfrac{7}{x-8}$. And this is the answer to the given example. Now I'll give another example.

Example 4

According to Stroud and Booth (2013)*, "Express each of the following in partial fractions: $\dfrac{5x^2-77}{x^2-2x-15}$."

Solution

The given expression is $\dfrac{5x^2-77}{x^2-2x-15}$. Since both the numerator and the denominator has the degree 2, this is an equal degree expression. Now in order to write this expression in partial fraction form, I have to remove both x and x^2 from the top. So I can write this expression as

$$\begin{aligned}\frac{5x^2-77}{x^2-2x-15} &= \frac{5(x^2-2x-15)+10x-2}{x^2-2x-15}\\ &= \frac{5(x^2-2x-15)}{x^2-2x-15}+\frac{10x-2}{x^2-2x-15}\\ &= 5+\frac{10x-2}{x^2-2x-15}.\end{aligned}$$

Now I have two components of this expression: one is 5 and the other is $\dfrac{10x-2}{x^2-2x-15}$. I cannot decompose 5 further. But I can still break down $\dfrac{10x-2}{x^2-2x-15}$. First I'll factorize $(x^2-2x-15)$. Thus it will be

$$\begin{aligned}x^2-2x-15 &= x^2+3x-5x-15\\ &= x(x+3)-5(x+3)\\ &= (x+3)(x-5).\end{aligned}$$

Next, I'll rewrite $\dfrac{10x-2}{x^2-2x-15}$ as $\dfrac{10x-2}{(x+3)(x-5)}$. Let me assume

$$\frac{10x-2}{(x+3)(x-5)} = \frac{A}{x+3} + \frac{B}{x-5}$$

where A, B are constants. Now my task is to find out the values of A and B.In order to do that, I'll simplify this expression as

$$\begin{aligned}\frac{10x-2}{(x+3)(x-5)} &= A(x-5)+B(x+3)(x+3)(x-5) \\ &= \frac{(A+B)x+(-5A+3B)}{(x+3)(x-5)}.\end{aligned}$$

Now I can compare coefficients on both sides. And I get two different equations as

$$A+B=10 \tag{5}$$

and

$$-5A+3B=-2. \tag{6}$$

Next, I'll solve equations (5) and (6) to get the values of A and B. Now I'll multiply equation (5) with 5 and add it to the equation (6). And this gives

$$B=6.$$

Next, I'll substitute this value of B in equation (5) to get

$$A=4.$$

Therefore the partial fraction of $\dfrac{10x-2}{(x+3)(x-5)}$ is $\dfrac{4}{x+3}+\dfrac{6}{x-5}$. Hence I can say that the partial fraction form of $\dfrac{5x^2-77}{x^2-2x-15}$ is $5+\dfrac{4}{x+3}+\dfrac{6}{x-5}$. And this is the answer to the given example.

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*Reference: K. A. Stroud and Dexter J. Booth (2013): Engineering mathematics, Industrial Press, Inc.; 7th Edition (March 8, 2013), Chapter: Partial fractions, Further problems F.8, p. 240, Q. No. 18 (Example 3), Q. No. 19 (Example 4).